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Critical behaviour of the continuous n-component Potts model†

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Abstract. The continuous n-component Potts model is studied in the framework of Wilson's multiplicative renormalization group. Apart from the isotropic ϕ^4 interaction, there is another one inherent in the model for $n > 2$. For $n \geqslant 4$, it is distinct from any ϕ^4 interaction that has been investigated in detail. Critical exponents are calculated. The n dependence of the fixed points shows new behaviour; in particular for n in the neighbourhood of five, the ϵ expansion must be reformulated as a power series in $\epsilon^{1/2}$.

1. Introduction

Recently interest in the Potts (1952) model has been greatly revived, largely due to the controversy over the nature of the phase transition. A first analysis using Landau theory (see eg Landau and Lifshitz 1958) or mean-field theory (Mittag and Stephen 1974) suggests that the model will exhibit a first-order phase transition, essentially because the model permits a coupling trilinear in the magnetization variable. This conclusion is independent of the dimension of space d.

Progress beyond mean-field theory has been made in three main directions. Firstly, in two dimensions Baxter (1973) showed that the Potts model on a square lattice has a first-order transition for $n > 3$. (We represent the lattice Potts model by an n-dimensional spin vector which can take $n+1$ states; $n = q-1$ in the more conventional notation.) For $n \leqslant 3$, the transition is of higher order. A first-order transition is expected for n large in all dimensions (R J Baxter, private communication).

Secondly, there has been considerable numerical work with series expansions, particularly for $n = 2$, $d = 3$ (Kihara *et al* 1954, Straley 1974, Enting 1974; see also Domb 1974 and references therein). The outcome of this analysis is still not entirely clear; if the system has a first-order transition for $n = 2$, $d = 3$, it must be a very small one.

Thirdly, one can formulate a continuum Potts model in terms of a spin density field $\phi_i(x)$. It is far from clear that the lattice and field models will have the same phase transition behaviour, but the latter are of some interest in their own right, and renormalization group methods may be readily applied to them. Golner (1973) uses the approximate recursion formula to show that the $n = 2$, $d = 3$ field model has a first-order transition. The exponent of the trilinear coupling at the Heisenberg fixed point has been calculated in the ϵ expansion (Wegner 1972, Amit and Shcherbakov 1974 ($n = 2$), Wallace and Zia 1975a) and $1/n$ expansion (Wallace and Zia 1975a) and all indications

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are that the trilinear coupling is relevant and produces a first-order transition in three dimensions. This view is disputed by Alexander (1974).

In this paper we generalize the analysis of Amit and Shcherbakov to all n and higher order in $\epsilon(\equiv 4-d)$. For $n > 2$ there appears a non-isotropic ϕ^4 interaction. For $n \geqslant 4$, this interaction, together with the isotropic one, provides an excellent example of a multiplicatively renormalizable system *distinct* from the well known isotropic–hypercubic one (Wilson and Fisher 1972, Wallace 1973, Aharony 1973, Cowley and Bruce 1973). The effect of the cubic perturbation on this 'new' system is also analysed.

As a prelude to our analysis, we briefly review a representation of the Potts n-component lattice model. A generalization of the Ising model, the Potts model consists of $n+1$ spin states on each lattice site with a nearest-neighbour interaction such that the energy is one value if the two nearest-neighbour states are different and another if they are the same. The spin states can be represented by a set of $n+1$ vectors in n space: e_i^α; $\alpha = 1, \dots n+1$, $i = 1, \dots n$, such that (repeated indices are summed unless specified otherwise)

$$e_i^\alpha e_i^\beta = \frac{n+1}{n}\delta^{\alpha\beta} - \frac{1}{n}. \tag{1}$$

In (1) we have chosen the normalization of the vectors to be unity for convenience. The nearest-neighbour interaction term can now be written as proportional to the scalar product of their spin states.

This set of e's is in fact the set of vectors defining the $n+1$ vertices of a hypertetrahedron in n-dimensional space. Although we will not need an explicit form for this set of vectors, we will provide an example by recursion on n. Let s^α be such a set of n vectors in $n-1$ space; then the $n+1$ vectors in n space are given by

$$\begin{aligned} e_i^\alpha &= \sin\theta_n s_i^\alpha && \text{for } \alpha = 1, \dots, n; i = 1, \dots n-1, \\ e_n^\alpha &= \cos\theta_n && \text{for } \alpha = 1, \dots, n, \\ e_i^{n+1} &= \delta_{n,i}, \end{aligned} \tag{2}$$

with $\cos\theta_n = -1/n$. The recursion may be started with $n = 1$, which is the Ising model.

In our analysis we will need two simple mathematical identities:

Lemma 1. $$\sum_\alpha e_i^\alpha = 0. \tag{3}$$

Lemma 2. $$e_i^\alpha e_j^\alpha = \frac{n+1}{n}\delta_{ij}. \tag{4}$$

To prove lemma 1, consider the magnitude of the vector in question:

$$\sum_\alpha e_i^\alpha \sum_\beta e_i^\beta = \sum_{\alpha,\beta}\left(\frac{n+1}{n}\delta^{\alpha\beta} - \frac{1}{n}\right) = \frac{n+1}{n}(n+1) - \frac{1}{n}(n+1)^2 \equiv 0.$$

That any n of the entire set form a complete set in n space is obvious; so any vector A_i may be written as $\Sigma_{\beta=1}^n a_\beta e_i^\beta$. Now consider

$$e_i^\alpha e_j^\alpha A_j = \sum_{\alpha,\beta} e_i^\alpha(a_\beta)\left(\frac{n+1}{n}\delta^{\alpha\beta} - \frac{1}{n}\right) = \frac{n+1}{n}a_\beta e_i^\beta = \frac{n+1}{n}A_i,$$

where we have used lemma 1. Equation (4) is established since A_i is any vector.

We now leave the lattice model and consider the 'continuous' (or field) model which we derive in § 2 and study in § 3 in the ϵ expansion. In § 4 we shall discuss the implications of this work for the phase transition behaviour of the system.

2. The continuum Potts model

The partition function of the lattice Potts model discussed in § 1 may be written as

$$Z(J) = \sum_{\{s\}} \exp(-\tfrac{1}{2}s_{i,\sigma}K_{\sigma\tau}s_{i,\tau} + J_{i,\sigma}s_{i,\sigma}). \qquad (5)$$

The indices σ and τ are summed over all lattice sites, the index i runs over the n-components of the spin variable s, and the configuration sum (denoted by $\{s\}$) runs over the $n+1$ unit vectors e_i^α for the spin at each lattice site. If there are only nearest-neighbour interactions, the symmetric coupling matrix $K_{\sigma\tau}$ vanishes unless σ and τ are nearest-neighbour lattice sites. Appropriate derivatives of $Z(J)$ generate all correlation functions.

Although much has been gleaned about the phase transition behaviour of this lattice model (see particularly Baxter 1973), much remains to be established; in this paper we study the ϕ^4 field theory obtained from (5) by a well known method (Baker 1962, Siegert 1963, Hubbard 1972). In this method we introduce a new variable $\phi_{i,\sigma}$ with n components for each lattice site σ and make the replacement

$$\exp(-\tfrac{1}{2}s_{i,\sigma}K_{\sigma\tau}s_{i,\tau}) = C\left(\prod_{j,\rho}\int \mathrm{d}\phi_{j,\rho}\right)\exp[\tfrac{1}{2}\phi_{i,\sigma}(K^{-1})_{\sigma\tau}\phi_{i,\tau} + \phi_{i,\tau}s_{i,\tau}]. \qquad (6)$$

Here K^{-1} is the inverse of the coupling matrix K and C is a constant; the identity is trivial to obtain by completing the square in the exponential on the right-hand side. The configuration sum in (5) is easy to perform because now the spins s at each lattice site are decoupled. The result may be written as

$$\sum_{\{s\}} \exp(\phi_{i,\sigma} + J_{i,\sigma})s_{i,\sigma} = \exp[-V(\phi+J)] \qquad (7)$$

where

$$V(\phi) = -\sum_{\sigma} \ln\left(\sum_{p=0}^{\infty} \frac{1}{p!}\phi_{i_1,\sigma}\dots\phi_{i_p,\sigma}v_{i_1\dots i_p}\right) \qquad (8)$$

and

$$v_{i_1\dots i_p} = \sum_{\alpha} e_{i_1}^{\alpha}\dots e_{i_p}^{\alpha}. \qquad (9)$$

In the continuum limit the variable $\phi_{i,\sigma}$ becomes a field $\phi_i(x)$, the multiple integral in equation (6) becomes a functional integral and the term $\phi K^{-1}\phi$ in equation (6) is written in terms of local derivatives of the field ϕ: ϕ^2, $\phi\nabla^2\phi$, $\phi\nabla^4\phi$ etc. $V(\phi+J)$ represents a local non-polynomial interaction of the field ϕ with a source field $J_i(x)$.

The advantage of transforming to the field variable ϕ, as far as phase transition behaviour is concerned, is that one hopes readily to identify a small number of relevant couplings (for reviews see Wilson and Kogut 1974, Ma 1973, Zinn-Justin 1974). Roughly speaking, when the functional integral is calculated by Feynman graph expansion, the high powers of the field produce graphs which have well behaved low-momentum behaviour and hence do not change the infrared singularities which generate critical exponents differing from mean-field theory. This credo is confirmed indirectly by the agreement between, eg, the ϵ expansion and high-temperature series expansions (some results are quoted in Wilson and Kogut 1974) and explicitly to a certain extent in the ϵ expansion (Wegner 1972, Wallace and Zia 1975a).

In this spirit we are led to consider the Euclidean field theory with a Hamiltonian

$$\frac{\mathscr{H}}{kT} = \int d^d x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} r_0 \phi^2 + \frac{1}{3!} q_0 Q_{ijk}\phi_i\phi_j\phi_k + \frac{1}{4!}(u_0 S_{ijkl} + f_0 F_{ijkl})\phi_i\phi_j\phi_k\phi_l \right) \tag{10}$$

where r_0 is linearly increasing with temperature and some appropriate momentum cut-off is understood, to reproduce the effect of the lattice in the original lattice model. Q and F are couplings of the form (9):

$$Q_{ijk} = \sum_\alpha e_i^\alpha e_j^\alpha e_k^\alpha, \tag{11a}$$

$$F_{ijkl} = \sum_\alpha e_i^\alpha e_j^\alpha e_k^\alpha e_l^\alpha, \tag{11b}$$

and S_{ijkl} is the symmetric coupling,

$$S_{ijkl} = \tfrac{1}{3}(\delta_{ij}\delta_{kl} + 2 \text{ permutations}). \tag{12}$$

Although the approximations applied in obtaining equation (10) from expression (5) involve essentially only the neglect of what are expected to be irrelevant couplings, they certainly do not guarantee equivalent phase transition behaviour for the two models. In particular the discrepancies apparent in the two-component model (see § 1 for references) may simply be due to the high powers of the field (which are neglected in (10)) producing an effective ϕ^3 coupling with a very small coefficient.

With this admonition let us return to expression (10) to study it as a field theory in its own right. The cases $n = 1$ (Ising model) and $n = 2$ (two-component Potts model) are familiar. In both cases the only ϕ^4 interaction is the symmetric one $(\phi^2)^2$. In the case $n = 3$, the vertices of the tetrahedron also form the body diagonals $(1, 1, 1)$ $(-1, -1, 1)$, $(-1, 1, -1)$ and $(1, -1, -1)$ of a cube (only half of the total number so this is not a decoupled Ising model—see Syozi 1972 p 325, and references therein) and the interactions can be represented by $\phi_1\phi_2\phi_3$, $(\phi^2)^2$ and $\phi_1^4 + \phi_2^4 + \phi_3^4$. Apart from the trilinear interaction, this is the well known symmetric/cubic system. For $n \geqslant 4$, the model does not correspond to any other known to us; we consider the renormalization of the ϕ^4 couplings and the effect of the trilinear coupling in § 3.

3. Critical exponents of the Potts model

We shall first consider the restricted Potts model by which we mean the system defined by the Hamiltonian (10) with no trilinear coupling ($q_0 = 0$). (It corresponds to a lattice model with $2(n+1)$ state vectors $\pm e_i^\alpha$.) To study the critical behaviour of such a system, the renormalization programmes of Wilson (1972) and Nickel (1975) or Brézin *et al* (1973) and Zinn-Justin (1974) could be followed; we choose the former. Although many general features of multi-component systems of this genre are known (Brézin *et al* 1974, Wallace and Zia 1974), there are interesting new features specific to the Potts model.

First, we must show that our system is multiplicatively renormalizable, that is, we need to demonstrate

$$\Gamma^{(2)}_{ij} \propto \delta_{ij}, \tag{13a}$$

$$\Gamma^{(4)}_{ijkl} = \text{linear combination of } S_{ijkl} \text{ and } F_{ijkl} \tag{13b}$$

where $\Gamma^{(p)}$ is a p-spin proper vertex function. The Hamiltonian being given by (10), both of these vertex functions must be formed by 'contractions' on all products of the three tensors δ_{ij}, S_{ijkl} and F_{ijkl}. Since S is formed from two δ's, and F from contracting the Greek indices of four e's (11), the most general expression for $\Gamma^{(p)}$ is a series of contractions (on both Latin and Greek indices) of δ's and an even number of e's. It is clear that any contraction reduces the number of e's by zero or two. Therefore, the final result for $\Gamma^{(2)}$ must be a linear combination of δ_{ij}, $e^{\alpha}_i e^{\alpha}_j$ and $\Sigma_\alpha e^{\alpha}_i \Sigma_\beta e^{\beta}_j$ which, by the lemmas, satisfies (13a). The same argument applies to $\Gamma^{(4)}$, the list of possible combinations consisting now of $\delta\delta$, δee and $eeee$. The first two always produce $\delta\delta$ (or vanish) while the last set will be nonzero if and only if there is one common Greek index (summed) or two pairs (summed); so there will be either an F or a $\delta\delta$. If we keep in mind that Latin indices must be symmetrized at the end to bring an 'asymmetric' combination of $\delta\delta$ into S, we see that (13b) is also satisfied.

To obtain critical exponents, we follow standard analyses. For completeness, we outline the plan:

(a) To (10), add the counter term $\frac{1}{2}r\phi^2$ and treat $\frac{1}{2}(r_0 - r)\phi^2 + \phi^4$ terms as a perturbation. Because of (13a), the susceptibility tensor χ_{ij} is of the form $\chi\delta_{ij}$; choose $r = \chi^{-1}$ so that $r \to 0_+$ is identified with the critical region.

(b) Find special values (u^*, f^*) of u and f to order ϵ for which the vertex functions scale in the limit $r \to 0$:

$$\lim_{r \to 0} \Gamma^{(4)}(\boldsymbol{k}_i = 0, r) \propto r^{\epsilon/2} \tag{14}$$

where $u \equiv u_0(2^d \pi^{d/2}\Gamma(d/2))^{-1}$ and similarly for f (cf Nickel 1975 equation (2.11)).

(c) For each of these, determine η at $O(\epsilon^2)$ via

$$\lim_{\boldsymbol{k} \to 0} \Gamma^{(2)}(\boldsymbol{k}, r = 0) \propto |\boldsymbol{k}|^{2-\eta}. \tag{15}$$

(d) Use the scaling law

$$\Gamma^{(4)}(0, r) \propto r^{(\epsilon - 2\eta)/(2-\eta)} \tag{16}$$

to determine u^* and f^* to $O(\epsilon^2)$.

(e) These are used to determine other exponents to order ϵ^2.

In principle, steps (c) and (d) can be repeated to obtain η and u^*, f^* to all orders, but we stop at ϵ^2.

Nickel (1975) has provided all the required analytic expressions. We only need to calculate the weights for our system, ie the tensorial contractions. (Note that Nickel defines his interaction term by $u_0\phi^4/4$; we choose 4! instead of 4. With this difference in mind, we can always check our results against his by setting f_0 to zero.) Defining

$$T_{ijkl} \equiv uS_{ijkl} + fF_{ijkl} \tag{17}$$

$$A \equiv (n+1)/n \tag{18a}$$

$$x \equiv (n+2)u/3 + Af \tag{18b}$$

$$y \equiv (n+8)u^2/9 + 2Auf/3 + (Af/n)^2 \tag{18c}$$

$$z \equiv 4uf/3 + A(2-A)f^2 \tag{18d}$$

$$w_1 \equiv 2u^2(x+2u)/9 + zA(u/3 + Af/n^2) \tag{18e}$$

$$w_2 \equiv 4u^2f/9 + z[2u/3 + A(A-2)f], \tag{18f}$$

we list the contractions corresponding to figures (1*a*), (1*b*), (1*c*), (1*d*) and (1*e*) respectively:

$$T_{ijkk} = x\,\delta_{ij} \tag{19a}$$

$$T_{ipqr}T_{pqrj} = \{xu + Af[u + (1+n^{-3})f]\}\,\delta_{ij} \tag{19b}$$

$$\{T_{ijpq}T_{pqkl}\} = yS_{ijkl} + zF_{ijkl} \tag{20a}$$

$$\{T_{ijpq}T_{pqrs}T_{rskl}\} = [x(y-2u^2/3) + w_1]S_{ijkl} + w_2F_{ijkl} \tag{20b}$$

$$\{T_{ijpq}T_{prsk}T_{qrsl}\} = [u(y-2u^2/3) + w_1]S_{ijkl} + [f(y-2u^2/3) + w_2]F_{ijkl}. \tag{20c}$$

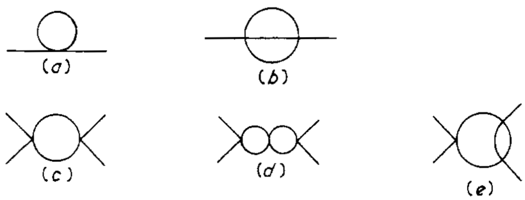

Figure 1. One- and two-loop graphs for $\Gamma^{(2)}$ and $\Gamma^{(4)}$.

The brackets { } in equations (20*a*, *b*, *c*) denote symmetrization in the indices *ijkl*. In addition to the contractions, the contributions of these graphs to the correlation functions include 4!'s and combinatorial factors: $\frac{1}{2}$, $\frac{1}{6}$, $\frac{3}{2}$, $\frac{3}{4}$ and 3 respectively.

Proceeding to step (*b*), we easily find the three sets of non-trivial (u^*, f^*) for which (14) holds:

(i) $f^* = 0,$ $\quad u^* = 3\epsilon/(n+8)$ (21*a*)

(ii) $f^* = n(n-4)\epsilon/3A(n^2-5n+8),$ $\quad u^* = \epsilon/(n^2-5n+8)$ (21*b*)

(iii) $f^* = n\epsilon/3A(n+3),$ $\quad u^* = \epsilon/(n+3).$ (21*c*)

As is well known, each of these special values—fixed points of a renormalization group transformation (Wilson 1972, Zinn-Justin 1974)—describes a different critical behaviour, in general. Labelling these respectively by Heisenberg (H) and Potts (P_1, P_2), we briefly discuss the results before continuing to step (*c*).

For $n = 2\,(3)$, it can be checked that the isotropic (isotropic–cubic) systems are recovered. For $n = 4$, from general arguments (Brézin *et al* 1974) H is degenerate (with P_1 in this case). For $n = 5$, P_1 and P_2 are degenerate. Next, we investigate the scaling behaviour at this order in ϵ. $\Gamma^{(4)}(T^*)$, where $T^* \equiv u^*S + f^*F$, scales as $r^{\epsilon/2}$; this is guaranteed by the choice of u^* and f^*. For each of these special values we can find another (scaling) variable,

$$\tilde{T} \equiv \tilde{u}S + \tilde{f}F, \tag{22}$$

with the property of being an eigenperturbation and scaling as $r^{w+\epsilon/2}$, ie

$$\Gamma^{(4)}(T^* + a\tilde{T}) \propto r^{\epsilon/2}(T^* + a'\tilde{T}r^{w}) + O(a^2) \tag{23}$$

where a is a small parameter and $a' \propto a$. (We emphasize that care is required in writing down the structure of corrections at higher orders in ϵ; one should use the general solution of the renormalization group equation, see eg Zinn-Justin (1974).)

If $w > 0$, the system is said to be stable against the perturbation. Otherwise, as $r \to 0$, the perturbation, no matter how small, eventually wins and the system crosses over to new behaviour. For each of (21), we list the eigenperturbations (in terms of $\tilde{u}/\tilde{f}$) and $\bar{w}$ (defined by $w = \bar{w}\epsilon + O(\epsilon^2)$) respectively:

$$\text{(H)} \qquad -3A/(n+2); \qquad (4-n)[2(n+8)]^{-1} \tag{24a}$$

$$(\text{P}_1) \qquad A(2A-3); \qquad (4-n)(n-5)[6(n^2-5n+8)]^{-1} \tag{24b}$$

$$(\text{P}_2) \qquad -A^2/2; \qquad (n-5)[6(n+3)]^{-1}. \tag{24c}$$

The first of these results has been discussed (Brezin *et al* 1974) in general terms. As usual, we simply remark that degeneracy (at this order) of the special values (cf (21)) is related to vanishing w. $O(\epsilon^2)$ contributions, in general, split such degeneracies and introduce more complicated behaviour.

The calculation of η for each of these fixed points is straightforward. In general, $\eta = (\epsilon^2 - 4w^2)/48 + O(\epsilon^3)$, and we list the coefficient of the ϵ^2 term:

$$\text{(H)} \qquad (n+2)[2(n+8)^2]^{-1} \tag{25a}$$

$$(\text{P}_1) \qquad (n-1)(n-2)(n^2-6n+11)[54(n^2-5n+8)^2]^{-1} \tag{25b}$$

$$(\text{P}_2) \qquad (n+1)(n+7)[54(n+3)^2]^{-1}. \tag{25c}$$

The next stage, (*d*), though mathematically straightforward, produces results that require care in interpretation. Introducing the more compact notation g_μ for the couplings:

$$g_1 \equiv u \qquad g_2 \equiv f, \tag{26}$$

we write

$$g^*_\mu = h_\mu\epsilon + \sigma_\mu\epsilon^2 + O(\epsilon^3). \tag{27}$$

The first-order special values h_μ can be read off from (21). For each of them we have checked that the coefficient of the $\epsilon^3 \ln^2 r$ term in equation (16) is indeed the correct one ($\frac{1}{8}$) for exponentiation. The coefficient of the $\epsilon^3 \ln r$ term provides a linear equation for σ_μ:

$$R^{\nu}_{\mu}\sigma_\nu + Ch_\mu + \tilde{C}\tilde{h}_\mu = 0 \tag{28}$$

where $\tilde{h}_\mu \propto (\tilde{u}, \tilde{f})$ of (22), say $\tilde{h}_1 \equiv \tilde{u}/\tilde{f}$ of (24) and $\tilde{h}_2 \equiv 1$; $\tilde{C}$ is a function of n coming purely from the decomposition of (20*b*) (evaluated at $g^* = h$) into h and $\tilde{h}$; C is the sum of $(\eta/\epsilon^2 - \frac{1}{4})$ and the rest of the (20*b*) decomposition; and R^{ν}_{μ} is defined via $M^{\nu\lambda}_{\mu}$ through (20*a*):

$$\tfrac{3}{2}\{TT\} = (M^{\nu\lambda}_{1}S + M^{\nu\lambda}_{2}F)g_\nu g_\lambda \tag{20a}$$

$$R^{\nu}_{\mu} \equiv 2M^{\nu\lambda}_{\mu}h_\lambda - \tfrac{1}{2}\delta^{\nu}_{\mu}. \tag{29}$$

Equation (23) implies that h and $\tilde{h}$ are eigenvectors of R with eigenvalues $\frac{1}{2}$ and $\bar{w}$ respectively. The solution to (28) is, therefore,

$$\sigma_\mu = -2Ch_\mu - (\tilde{C}/\bar{w})\tilde{h}_\mu. \tag{30}$$

Taking into account the weight ($\frac{3}{4}$) of (20b) and a factor 2 from the analytic expression, we give C and $\tilde{C}$ in terms of h and $\tilde{h}$:

$$C = [B - h_1(1-2h_1) - \tfrac{1}{4}\bar{w}^2 - \tfrac{3}{16}]/3 \tag{31a}$$

$$\tilde{C} = -Bh_2/3\tilde{h}_2 \tag{31b}$$

where

$$B \equiv (x^*/\epsilon)(1-2h_1)\frac{h_1\tilde{h}_2}{h_1\tilde{h}_2 - h_2\tilde{h}_1} \tag{32}$$

In these expressions, $\bar{w}$ is to be taken from (24) and x^* is the fixed-point value of x in equation (18b) at order ϵ. For the three sets of special values, we evaluate B explicitly:

$$\text{(H)} \qquad (n+2)^2(n+8)^{-2} \tag{33a}$$

$$(\mathrm{P}_1) \qquad (n-1)(n-2)^2(n-3)(n^2-6n+11)^{-1}(n^2-5n+8)^{-2} \tag{33b}$$

$$(\mathrm{P}_2) \qquad 4(n+1)^2(n+7)^{-1}(n+3)^{-2} \tag{33c}$$

Since $\bar{w}$ vanishes for certain values of n it is clear that some singularities may occur in equation (30). For the $n = 4$ case, the problem does not arise, since $\tilde{C}$ vanishes either identically, (H), or as $(n-4)$, (P_1). The $n = 5$ singularity, however, is genuine. We will discuss this problem in § 4; for now, it suffices to take the point of view that equation (27) is invalid if $\sigma_\mu \sim \mathrm{O}(1/\epsilon)$.

To complete this section, we will use equation (30) to find the exponent γ and an exponent associated with the trilinear coupling to order ϵ^2. Following Nickel (1975) once more, we employ the equation

$$\Gamma_{2s}(\boldsymbol{k} = 0, r) \propto r^{1-1/\gamma}. \tag{34}$$

The weights and contractions corresponding to figures (1c), (1d) and (1e) are, respectively,

$$x/2, \qquad x^2/4, \qquad \tfrac{1}{2}[y + Az + 2(n-1)u^2/9] \tag{35}$$

where A, x, y, z are defined in (18). After checking exponentiation, we arrive at

$$\frac{1}{\gamma} = 1 - \frac{\epsilon}{2}\left\{\left(\frac{x^*}{\epsilon}\right) + \frac{\epsilon}{2}\left[2\left(\frac{n+2}{3}\sigma_1 + \frac{n+1}{n}\sigma_2\right) + \left(\frac{x^*}{\epsilon}\right)^2\right]\right\} + \mathrm{O}(\epsilon^3). \tag{36}$$

As in equation (32), x^* is the value of expression (18b) at any fixed point *to order* ϵ.

Finally we ask how the restricted Potts model is perturbed by the trilinear coupling $q_0 Q_{ijk}\phi_i\phi_j\phi_k$ in equation (10). The effect of this perturbation is governed by the exponent ψ associated with the three-point function $\Gamma^{(3)}_{ijk}(\{\boldsymbol{k}\}, r)$ linear in q_0:

$$\Gamma^{(3)}_{ijk}(0, r) \propto q_0 r^\psi$$

in the critical region $r \to 0$. If the contribution of $\Gamma^{(3)}$ in the effective potential

$$F(M, r) = \sum \frac{1}{p!} M_{i_1} \dots M_{i_p} \Gamma^{(p)}_{i_1 \dots i_p}(0, r)$$

is more singular than the terms with $q_0 = 0$, eg $M^2\Gamma^{(2)}(\equiv M^2 r)$, then q_0 is a relevant coupling which perturbs the system away from the appropriate fixed point of the restricted model, presumably into a first-order phase transition.

The exponent ψ is obtained in the usual way by exponentiating logarithms in the graphs contributing to $\Gamma^{(3)}$. The analytic expressions are identical to the ones used for $\Gamma^{(4)}$; only a set of weights and contractions is needed. Corresponding to figures 2(*a–d*), with the common factor q_0 suppressed, these are:

$$\tilde{x}, \qquad \tilde{x}^2/3, \qquad \tfrac{3}{2}[y+A(2-A)z]-\tfrac{1}{3}u^2, \qquad 2\tilde{x}^2/3 \tag{37}$$

where $\tilde{x} \equiv u+3A(2-A)f/2$. The result may be written as

$$\psi = \bar{\psi}\epsilon+(\sigma_1+\tfrac{3}{2}A(2-A)\sigma_2-\tfrac{1}{3}\bar{\psi}^2)\epsilon^2+O(\epsilon^3) \tag{38}$$

where σ_1 and σ_2 are given for any fixed point in equations (30) to (32), and $\bar{\psi}$ is given for the three fixed points by

$$\text{(H)} \qquad \bar{\psi} = 3(n+8)^{-1} \tag{39a}$$

$$(\text{P}_1) \qquad \bar{\psi} = \tfrac{1}{2}(n-2)(n-3)(n^2-5n+8)^{-1} \tag{39b}$$

$$(\text{P}_2) \qquad \bar{\psi} = \tfrac{1}{2}(n+1)(n+3)^{-1} \tag{39c}$$

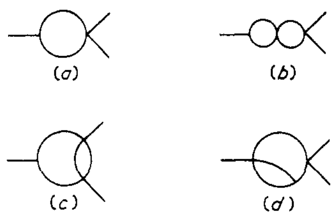

Figure 2. One- and two-loop graphs for $\Gamma^{(3)}$.

Note that for $n = 2$, the interaction $F\phi^4$ is proportional to the isotropic term $(\phi^2)^2$, and only ψ(H) is meaningful. The expression (39*a*) agrees with the results of Amit and Shcherbakov (1974) and is a special case contained in Wallace and Zia (1975a). Discussion of the effect of $Q\phi^3$ on phase transition behaviour is deferred until § 4.

4. Discussion and conclusion

We have generalized the analysis of Amit and Shcherbakov (1974) and obtained the effect of the trilinear coupling for the n-component Potts model in terms of the exponent ψ (equation (38)). To interpret this result, we must first study the model without this coupling (the restricted Potts model). There, unlike the two-component case, we found that two ϕ^4 couplings, u and f, are necessary to describe the general system. Given n, the completely stable one of the three fixed points (H, P_1 and P_2) in the u–f plane will control the $q_0 = 0$ model. It is about this point that we should consider the effect of $Q\phi^3$ through ψ. So we first examine the configuration of the three fixed points and their variation with n—in fact, that in its own right is of intrinsic interest and provides an example of the behaviour generic in the sense of catastrophe theory. (An elementary introduction and references are given by D R J Chillingworth in a 1973 Southampton University Mathematics Department preprint, *Elementary Catastrophe Theory*).

The fixed-point configuration and its variation with n is best visualized via the renormalization group potential function $V(u, f)$ (Wallace and Zia 1974, 1975b), of which the fixed points are critical points. To lowest non-trivial order, the qualitative features of the system are summarized by: a very strongly n-dependent fixed point (P_1) and two relatively weakly n-dependent ones (H, P_2). Defining n_c to be values of n at which two fixed points become degenerate, we have, at this order, the following picture:

$$n_c(H) = 4 \qquad P_1 = H$$

$$n_c(P_+) = n_c(P_-) = 5 \qquad P_1 = P_2$$

(the necessity for $P_\pm$ will become clear later). This variation with n is qualitatively shown in figure 3. A certain section of V is shown on figure 4 for various regions of n.

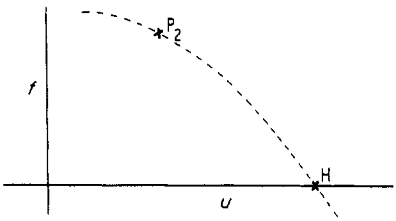

Figure 3. Fixed points in the u–f plane. The broken curve gives the qualitative n dependence of (P_1) at order ϵ, the movement of (H) and (P_2) with n is neglected for clarity.

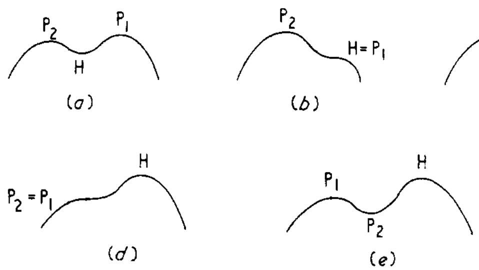

Figure 4. Qualitative form of tree and one-loop contributions to the renormalization group potential function V in a section along the broken curve of figure 3 for (*a*) $n < 4$, (*b*) $n = 4$, (*c*) $4 < n < 5$, (*d*) $n = 5$, (*e*) $n > 5$.

For $n \sim 4$, the potential function around the (most) stable fixed point behaves as

$$V \sim -\rho^3 + (n-4)\rho^2 \tag{40}$$

where ρ is a measure of the 'distance' from the stable fixed point. A *generic* form (in catastrophe theory language), on the other hand, is

$$V \sim -\rho^3 + \kappa\rho \tag{41}$$

which describes critical points colliding and disappearing as κ decreases through zero. If we cast (40) into the generic form by a displacement, we find that $\kappa \propto (n-4)^2$ so that fixed points may collide, but never disappear. The $n \sim 5$ behaviour is essentially the same at this order, only the 'sign' of ρ changes.

For n close to 4 or 5, it is reasonable to expect, therefore, higher orders to become important. As it turns out, the fact that (H) is guaranteed to exist at all orders implies that all higher-order corrections modify essentially only the coefficient of the ρ^2 term in (40). The result is that an ϵ expansion may be obtained for n_c:

$$n_c = 4 - 2\epsilon + O(\epsilon^2) \tag{42}$$

and the behaviour remains non-generic.

No guarantee of this nature exists for the other fixed points and, in general, we expect corrections to (40) of the form ($\lambda, \lambda' \sim O(1)$ at $n = 5$)

$$V \sim \rho^3 + (n - 5 + \lambda'\epsilon)\rho^2 + \lambda\epsilon\rho. \tag{43}$$

Casting this expression into the generic form, we see that

$$\kappa \sim (n - 5 + \lambda'\epsilon)^2 + \lambda\epsilon. \tag{44}$$

If $\lambda > 0$, κ never vanishes and the critical points never become degenerate. However, if $\lambda < 0$, there is a region of n around 5 such that there will be no critical points:

$$|n - 5| < \sqrt{(-\lambda\epsilon)} + O(\epsilon) \tag{45}$$

so that the 'ϵ expansion' of the n_c is in fact in $\sqrt{\epsilon}$. Having naturally two roots, we label them separately:

$$n_c(P_\pm) = 5 \pm \sqrt{(12\epsilon)} + O(\epsilon). \tag{46}$$

Although this behaviour is more complicated, it is of generic form; the Heisenberg case is a very special one indeed. That λ in (45) is -12 can be obtained by substituting $n = 5 + n'\sqrt{\epsilon}$ into equations (27) and (30) for $P_{1,2}$ and demanding degeneracy.

To summarize, we have the following picture for the 'Potts degeneracy'. As n increases through $n_c(P_-)$, P_1 and P_2 collide and disappear. In the region $|n - 5| < \sqrt{(12\epsilon)}$ there are *no* completely stable fixed points in the u–f plane—presumably the system has a first-order phase transition even if $q_0 \equiv 0$. As n increases through $n_c(P_+)$, P_1 and P_2 reappear with P_2 being the completely stable fixed point.

Graphically, we picture in figure 5 the two-loop contributions to V in the region around $P_{1,2}$ for $n \sim 5$. Adding this to that of figure 4 produces figure 6 which summarizes the collision–disappearance–reappearance phenomenon of fixed points.

Figure 5. Qualitative form of the two-loop contribution to V for n in the neighbourhood of five. Only the region around the P_1–P_2 degeneracy of figure 4(*d*) is shown.

Since this order of perturbation theory has produced a potential function V for which the parameter n is a control variable in a universal unfolding of the ρ^3 form of the degenerate situation, no new qualitative features will appear at higher orders in ϵ. However, the quantitative effects of higher-order terms must be expected to be large, because the expansion (46) is clearly not reliable for $\epsilon = 1 (d = 3)$.

This conclusion makes us refrain from quoting values of the exponents ψ and γ as power series in ϵ for low values of n (or in $\sqrt{\epsilon}$ for n near five) and underlines the argument of Wilson that a quantitative description of the phase transition behaviour is more difficult when a system has several fixed points close to one another. On the other hand, for n large, the fixed-point structure is unambiguous and exponents may be quoted. In that

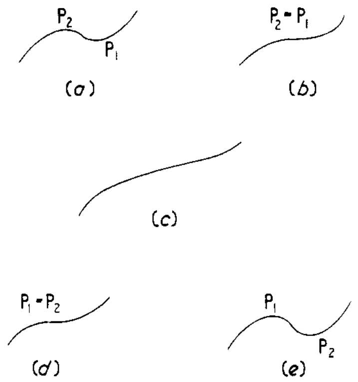

Figure 6. Combined effects of figures 4 and 5, illustrating the disappearance and reappearance of P_1 and P_2. (*a*) $n < n_c(P_-)$, (*b*) $n = n_c(P_-)$, (*c*) $n_c(P_-) < n < n_c(P_+)$, (*d*) $n = n_c(P_+)$, (*e*) $n > n_c(P_+)$.

case, P_2 is the controlling fixed point. For $d = 3$ (to lowest order in ϵ), η approaches the Ising value of $\frac{1}{54}$ while γ approaches $\frac{4}{3}$ (as against $\frac{7}{6}$ for Ising), showing again fundamental differences between the restricted Potts model and the hypercubic one. Finally, ψ becomes $\frac{1}{2}$ ($O(\epsilon^2)$ corrections are small) so that the $Q\phi^3$ term is indeed relevant, and its effect in $F(M, r)$ compared to the M^2r term diverges (as r vanishes) as $r^{-1/4}$. In this connection, we point out the significant difference between perturbing (with $Q\phi^3$) about H and P_2, the former leading to $\bar{\psi} \sim O(1/n)$ (equation (39*a*), which agrees with Oppermann 1975).

In conclusion, this analysis appears to favour a first-order transition for the standard Potts model, while for the restricted model such a transition is conjectured for a range of low *n*.

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